TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 264, Number 2, April 1981

CONFIGURATIONS OF SURFACES IN 4-MANIFOLDS[1]

BY

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ABSTRACT. We consider collections of surfaces $\{F_i\}$ smoothly embedded, except for a finite number of isolated singularities, self-intersections, and mutual intersections, in a 4-manifold M. A small 3-sphere about each exceptional point will intersect these surfaces in a link. If $[F_i] \in H_2(M)$ are linearly dependent modulo a prime power, we find lower bounds for Σ genus (F_i) in terms of the $[F_i]$, and invariants of the links that describe the exceptional points.

0. Introduction. The following special case of our main theorem is easy to state.

THEOREM 0.1. *Let M be a closed smooth 4-manifold and $\{F_i\}$ a collection of n smoothly embedded surfaces in general position. Let $x_i = [F_i] \in H_2(M)$. Suppose $\bigcup F_i$ is connected and $\sum a_i x_i = p^r y$ where p is a prime, $0 < a_i < p^r$, and $a_i \neq 0 \bmod p$. Let $\#$ be the total number of intersection points. Then*

$$\# + 2\sum \operatorname{genus}(F_i) \geqslant \left|2y\left(\sum x_i - y\right) - \sum_{i<j} x_i x_j - \operatorname{sign} M\right| + 2(n-1) - \dim H_2(M, \mathbf{Z}_p).$$

For example, according to a theorem of C. T. C. Wall [W] if M is a smooth closed simply-connected 4-manifold with indefinite quadratic form, then in $M \# S^2 \times S^2$ any primitive noncharacteristic class may be represented by an embedded 2-sphere. Let M be $S^2 \times S^2$ and let F_1 be a 2-sphere representing $(0, 1, 0, 0) \in H_2(S^2 \times S^2 \# S^2 \times S^2)$ (with respect to the natural basis). Let F_2 be a 2-sphere representing $(a, b, 0, 0)$ transverse to F_1 where $a > 1$, $b > 0$ and $(a, b) = 1$. Let $\#$ be the total number of intersection points of F_1 and F_2. Then we have

$$\# \geqslant \begin{cases} ab - 2 & \text{if } a \text{ is even}, \\ ab - \dfrac{a}{d}\left[\dfrac{b}{d}\right] - 2 & \text{if } \left[\dfrac{b}{d}\right] \neq b \bmod 2, \\ ab - \dfrac{a}{d}\left(\left[\dfrac{b}{d}\right] + 1\right) - 2 & \text{if } \left[\dfrac{b}{d}\right] = b \bmod 2, \end{cases}$$

where d is the largest odd prime power dividing a. To see this, if a is even, let $p^r = 2$ and apply Theorem 0.1. Otherwise choose

$$p^r = d, \quad a_1 = (b - [b/d]d + d)/2 \quad \text{or} \quad a_1 = (b - [b/d]d)/2$$

Received by the editors September 19, 1979.

AMS (MOS) subject classifications (1970). Primary 57R95, 57M25.

[1]This paper is a revised version of the author's doctoral dissertation at the University of California, Berkeley, May 1978.

(whichever is integral) and $a_2 = (d - 1)/2$. This is in general a much better bound than one gets by calculating the algebraic intersection: $\# \geqslant a$.

Theorem 0.1 for $n = 1$ and $H_1(M) = 0$ is a theorem of V. A. Rokhlin [**R**]. W. C. Hsiang and R. Szczarba [**H-S**] proved a similar result. That the $H_1(M) = 0$ hypothesis is unnecessary follows from V. I. Itenberg's higher dimension generalization of Rokhlin's results [$\mathbf{I_2}$] (see Corollary 2.4 of this paper). Our method of proof is similar to those above. However, we consider unbranched covers of the complement of a neighborhood of the surfaces instead of branched covers. Instead of using the G-Signature Theorem directly, we calculate $\sigma(L, \psi)$, a signature invariant of finite cyclic covers of 3-manifolds, for L the boundary of this neighborhood. $\sigma(L, \psi)$, which was first introduced by A. J. Casson and C. McA. Gordon, is basically a reformulation of the α-invariant defined by M. Atiyah and I. M. Singer.

I should mention that bounds similar to those given in Theorem 0.1 follow from Rokhlin's Theorem together with various ad hoc geometric arguments. In fact in the situation where

$$a_1 = a_2 = \cdots = a_n \tag{1}$$

and

$$x_i \cdot x_{i+1} \quad \text{for } 1 \leqslant i \leqslant n - 1 \tag{2}$$

and the quantity inside the absolute value sign all have the same sign, one can derive the same bounds. If not, all the a_i are equal; the bounds obtained cannot be stated in a simple general form. In many particular examples, the bounds obtained in this manner are significantly worse than those obtained from Theorem 0.1, and in no known example does one get better bounds. In the above example in $S^2 \times S^2 \# S^2 \times S^2$, one can derive the same bound for a even by this method. However, consider the family of examples given by $(a, b) = (2n + 1, 3n + 2)$ for $2n + 1$ a prime. By Theorem 0.1 $\# \geqslant 6n^2 + 7n - 2$. The best bound I can get using Rokhlin's theorem is $\# \geqslant 6n + 9 - 4/n$.

This paper is organized as follows. In §1, we give preliminary definitions and results concerning the homology of finite cyclic covers. In §2, we prove Theorem 2.1 which gives an obstruction to embedding a 4-manifold N with boundary L in a closed 4-manifold M. We then give a more precise definition of a configuration of surfaces, define the neighborhood of a configuration, and then specialize Theorem 2.1 to the case N is a neighborhood of a configuration. The obstruction involves $\sigma(L, \psi)$ and $\eta(L, \psi)$, a second invariant of these covers. Next we apply the same argument to higher dimensional codimension-0 embeddings. See Theorem 2.3. We conclude the section with a conjectured formula (2.5) for $\sigma(L, \psi)$ for certain (L, ψ).

In §3, we show how to calculate $\sigma(L, \psi)$ and $\eta(L, \psi)$ for any finite cyclic cover of a 3-manifold L. Our formula (3.6) generalizes a formula due to Casson and Gordon. A result of K. Murasugi and a result of A. G. Tristram fall out for free from (3.6). If L is described as the boundary of a neighborhood of a configuration of surfaces (and the cover satisfies a certain condition) we give formulas for $\sigma(L, \psi)$ and $\eta(L, \psi)$ in terms of the signature and nullity invariants of links associated to the links about each exceptional point and the homology classes given by the surfaces (3.7). In case L is the boundary of a plumbing, this formula is particularly simple. In a later paper, this simple formula will be used to calculate the Casson-Gordon invariants of 3-strand Turk's Head Knots.

In the fourth section, we combine the results of §§2 and 3 to give our main result (4.1). As a corollary, we derive the Tristram-Murasugi bounds for the slice genus of a link (4.3). This illustrates the well-known relation between Rokhlin's and Tristram's methods in a particularly vivid manner. We also derive as a corollary a theorem of O. Ya. Viro generalizing Rokhlin's result to the case of a single surface with a single singularity given as a cone on a knot. We then work out some explicit examples of applications of the main theorem.

In §5, we perform some calculations that make our results on 3- and 4-manifolds independent of the G-Signature Theorem. In the final section, we discuss the ad hoc geometrical constructions, mentioned above, that together with Rokhlin's Theorem give bounds of the type given by Theorem 0.1.

I would like to thank my advisor Professor Emery Thomas for much patient advice, guidance and encouragement. I am indebted to him for many fruitful ideas. Also I benefited greatly from learning Professor Robion Kirby's point of view on three and four dimensional manifolds. I thank him for his help.

We adopt the following conventions and definitions. All manifolds will be assumed smooth, oriented, and compact (unless they are described as an interior of a closed manifold). All other spaces (except $B\mathbf{Z}_d$) will be assumed to have the structure of a finite simplicial complex. The group $\mathbf{Z}_d$ will be thought of as the integers modulo d, with a specified generator, the residue class of one. Throughout ω will denote $e^{2\pi i/d}$, and p will be a prime number. We write $\beta_i(X)$ for $\dim H_i(X, Q)$ and $\rho_i(X)$ for $\dim H_i(X, \mathbf{Z}_p)$. The reduction of homology classes mod d will be indicated by ρ. Σ and $+$ will denote the disjoint union of spaces, as well as ordinary summation. rX will indicate the disjoint union of r copies of X. (a, d) denotes the g.c.d. of a and d. $l|d$ will mean l divides d.

1. Preliminaries on finite cyclic covers. Let $\mathbf{Z}_d$ act on a space Y, with the generator acting by $T\colon Y \to Y$. $H_k(Y, \mathbf{C})$ splits into a direct sum of eigenspaces $H_k(Y,j) = \{x | T_* x = \omega^j x\}$. Define $\beta_k(Y,j) = \dim H_k(Y,j)$ and $\bar{\beta}_k(Y) = \beta_k(Y, 1)$. Define $\chi(Y,j)$ to be $\Sigma(-1)^k \beta_k(Y,j)$ and $\bar{\chi}(Y) = \chi(Y, 1)$. Make analogous definitions for pairs (Y, Y').

If $\mathbf{Z}_d$ acts as a group of orientation preserving diffeomorphisms on a $2k$-manifold Y (possibly with boundary) define $\sigma_j(Y)$ to be the signature of the complexified intersection pairing restricted to $H_k(Y,j)$. This pairing is hermitian if k is even and skew hermitian if k is odd. The signature of a skew hermitian pairing $x \cdot y$ is defined to be the signature of the associated hermitian pairing given by $x * y = ix \cdot y$.

Isomorphism classes of cyclic d-fold covers of a fixed space X with specified generator T for the group of covering translations correspond bijectively to elements ψ of

$$[X, B\mathbf{Z}_d] = H^1(X, \mathbf{Z}_d) = \mathrm{Hom}(H_1(X), \mathbf{Z}_d)$$

since $B\mathbf{Z}_d$ is the classifying space for principal $\mathbf{Z}_d$-bundles. If ψ is thought of as a map $H_1(X) \to \mathbf{Z}_d$, then the element of $\mathbf{Z}_d$ defined by lifting a loop γ is $\psi[\gamma]$. We let X_ψ denote the covering space defined by ψ. Thus X_ψ comes equipped with a covering translation T. An element x of a $\mathbf{Z}_d$-module is called primitive if $x = ly$, $l \in \mathbf{Z}$, implies $(l, d) = 1$. $\psi \in H^1(X, \mathbf{Z}_d)$ is primitive if and only if ψ: $H_1(X) \to \mathbf{Z}_d$ is onto. X_ψ is connected if and only if X is connected and ψ is primitive.

If L is a $2k - 1$ manifold and $\psi \in H^1(L, \mathbf{Z}_d)$ then (L, ψ) represents an element in $\Omega_{2k-1}(B\mathbf{Z}_d)$. The bordism spectral sequence shows that $\Omega_*(\text{point}) \to \Omega_*(B\mathbf{Z}_d)$ is a rational isomorphism. Thus $\Omega_{2k-1}(B\mathbf{Z}_d)$ is torsion and $r(L, \psi) = \partial(W, \psi)$ for some $2k$-manifold W and integer r. Define $\sigma(L, \psi) = (\sigma_1(W_\psi) - \text{sign}\, W)/r$. Let N be a closed $2k$-manifold and (N, ψ) represent an element in $\Omega_{2k}(B\mathbf{Z}_d)$. By the above $r(N, \psi)$ is bordant to some $(N', 0)$ for some integer r. $\sigma_1(\ _\psi)$ and sign are both bordism invariants and $\sigma_1(N_0') = \text{sign}(N')$. It follows that $\sigma_1(N_\psi) = \text{sign}\, N$. Novikov additivity then shows $\sigma(L, \psi)$ is well defined. This is a straightforward generalization of the invariant defined by Casson and Gordon for $k = 2$ in **[C-G$_1$]** and **[C-G$_2$]**. Here we adopt the sign convention of the first paper which is opposite that of the second. Also define $\eta(L, \psi) = \bar{\beta}_{k-1}(L_\psi)$.

Let ψ' be a map $H_1(L) \to Q/\mathbf{Z}$. Pick d so the subgroup generated by $(1/d)$ and isomorphic to $\mathbf{Z}_d$ includes the image of ψ'. This defines a map ψ: $H_1(L) \to \mathbf{Z}_d$. One can show $\sigma(L, \psi)$ and $\eta(L, \psi)$ are independent of the choice of d. If $(s, d) = 1$, it is easy to see that $\sigma(L, s\psi) = \sigma_s(W_\psi) - \text{sign}\, W$ and $\eta(L, s\psi) = \beta_1(L_\psi, s)$. Using Lemma 7.4 of **[T-W]** together with the above observation concerning the independence of d, one sees these formulae hold even if $(s, d) \neq 1$.

PROPOSITION 1.1. $\bar{\chi}(X_\psi) = \chi(X_\psi, j) = \chi(X)$.

PROOF. A simplex counting argument shows that if X_l is any l-fold cover of X, $\chi(X_l) - \chi(X) = (l - 1)\chi(X)$. For any $l|d$, one has a quotient l-fold cover of X. E. Thomas and J. Wood analyze the relations between the different eigenspaces of this collection in **[T-W**, §7]. They consider the middle dimension of branched covers of manifolds but the arguments go through unchanged. Lemmas 7.2, 7.4 and 7.5 of **[T-W]** then give the desired result.

For the rest of this section d will be a power of p. We will need an exact sequence of Smith homology groups due to E. E. Floyd **[F]**. We only need this sequence for unbranched covers or, equivalently, free actions. In this situation, the concepts and arguments are considerably simpler. On the other hand, we need this sequence in slightly greater generality ((1.2) is proved for $d = p$ and $1 < m < d$ in **[F]**). For these reasons we outline the results we need.

Give X_ψ the simplicial structure obtained by lifting the simplices in X. Let $C(X_\psi)$ denote the simplicial chain group of X_ψ and $C(X_\psi, \mathbf{Z}_p)$ be the simplicial chain group with $\mathbf{Z}_p$ coefficients. Let $\delta = 1 - T_*$: $C(X_\psi, \mathbf{Z}_p) \to C(X_\psi, \mathbf{Z}_p)$, so $\delta^d = 1 - T_*^d = 0$ and $\delta^{d-1} = 1 + T_* + \cdots + T_*^{d-1}$. By considering the subspace generated by the simplices covering a single simplex one sees kernel δ = image δ^{d-1}. Then a simple induction argument shows kernel δ^m = image δ^{d-m} for $0 < m \leqslant d$. Define $C^{\delta^m}(X_\psi)$ = kernel δ^m for $0 < m \leqslant d$. Since δ is a chain map, $C^{\delta^m}(X_\psi)$ is a subcomplex. Define $H^{\delta^m}(X_\psi)$ to be the homology of this complex. Since $C(X_\psi, \mathbf{Z}_p) = C^{\delta^d}(X_\psi)$ and $C(X, \mathbf{Z}_p) = C^{\delta}(X_\psi)$, we have $H(X_\psi, \mathbf{Z}_p) = H^{\delta^d}(X_\psi)$ and $H(X, \mathbf{Z}_p) = H^{\delta}(X_\psi)$.

There is a short exact sequence of chain complexes:

$$0 \to C^{\delta}(X_\psi) \to C^{\delta^m}(X_\psi) \xrightarrow{\delta} C^{\delta^{m-1}}(X_\psi) \to 0$$

where the first map is inclusion and the second is given by $z \mapsto \delta(z)$. δ is onto because $\delta(\text{image } \delta^{d-m}) = \text{image } \delta^{d-m+1}$. There is a corresponding long exact sequence:

$$\to H_k^{\delta}(X_\psi) \to H_k^{\delta^m}(X_\psi) \to H_k^{\delta^{m-1}}(X_\psi) \to H_{k-1}^{\delta}(X_\psi) \to . \tag{1.2}$$

Proposition 1.3 below is a special case of a theorem attributed to Smith Theory, concerning branched covers, stated (without proof) by V. S. Itenberg **[I₂]**. He refers to O. Ya. Viro **[V₃]** for a special case which is still more general than (1.3).

PROPOSITION 1.3. $\beta_k(X_\psi) - \beta_k(X) \leqslant (d-1)\rho_k(X)$.

PROOF. Let μ: $C(X) \to C(X_\psi)$ denote the map which sends a simplex in X to the sum of simplices covering it. Let Γ be $C(X_\psi)$ modulo the image of μ. Γ is also a free chain complex. We have the following short exact sequence

$$0 \to C(X) \to C(X_\psi) \to \Gamma \to 0.$$

If we tensor this sequence with $\mathbf{Z}_p$, we will recover the sequence of Smith complexes for $m = d$. Thus $H(\Gamma, \mathbf{Z}_p) = H^{\delta^{d-1}}(X_\psi)$. Since μ_*: $H(X, Q) \to H(X_\psi, Q)$ is injective (the covering projection yields a left inverse), we have

$$\dim H_k(\Gamma, Q) = \beta_k(X_\psi) - \beta_k(X).$$

So by the universal coefficient theorem, $\dim H^{\delta^{d-1}}(X_\psi) \geqslant \beta_k(X_\psi) - \beta_k(X)$.

Finally since $\rho_k(X) = \dim H_k^{\delta}(X_\psi)$, induction using (1.2) shows $\dim H_n^{\delta^{d-1}}(X_\psi) \leqslant (d-1)\rho_k(X)$.

PROPOSITION 1.4. $\bar{\beta}_k(X_\psi) \leqslant \rho_k(X)$.

PROOF. Let X_s denote the quotient p^s-fold cover of X, $0 \leqslant s \leqslant r$. Lemmas (7.2) and (7.3) of **[T-W]** show

$$p^{r-1}(p-1)\bar{\beta}_k(X_r) = \beta_k(X_r) - \beta_k(X_{r-1}).$$

By (1.3) $\beta_k(X_r) - \beta_k(X_{r-1}) \leqslant (p-1)\rho_k(X_{r-1})$. Induction using (1.2) as above but applied to the cover $X_{r-1} \to X$ yields $\rho_k(X_{r-1}) \leqslant p^{r-1}\rho_k(X)$. The result follows.

L. Kaufman and L. Taylor essentially proved (1.5) below for $d = 2$ (Theorem (3.8) of **[K-T]**). Our proof is obtained by substituting an inductive argument using the Smith sequence above for the single application of the Gysin sequence of **[K-T]**.

PROPOSITION 1.5. *If X_ψ is connected, then $\bar{\beta}_1(X_\psi) \leqslant \rho_1(X) - 1$.*

PROOF. Let Y be a wedge of $\rho_1(X)$ circles and pick a simplicial map i: $Y \to X$ inducing an epimorphism $H_1(Y, \mathbf{Z}_d) \to H_1(X, \mathbf{Z}_d)$ and an isomorphism $H_1(Y, \mathbf{Z}_p) \to H_1(X, \mathbf{Z}_p)$. Pulling back the cover X_ψ to Y, we get a connected cover $\tilde{Y}$ of Y. If Y_s is a p^s fold connected cover of Y, then an Euler characteristic calculation shows

$\beta_1(Y_s) - \beta_1(Y_0) = (p^s - 1)(\rho_1(X) - 1)$. The Thomas-Wood argument then shows $\bar{\beta}_1(\tilde{Y}) = \rho_1(X) - 1$.

The exact sequence of Smith groups is natural, so we have

$$\begin{array}{ccccccc} H_1^{\delta}(\tilde{Y}) & \to & H_1^{\delta^m}(\tilde{Y}) & \to & H_1^{\delta^{m-1}}(\tilde{Y}) & \to & H_0^{\delta}(\tilde{Y}) \\ \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr \\ H_1^{\delta}(X_\psi) & \to & H_1^{\delta^m}(X_\psi) & \to & H_1^{\delta^{m-1}}(X_\psi) & \to & H_0^{\delta}(X_\psi) \end{array}$$

The five lemma permits one to prove inductively that $H_1^{\delta^m}(\tilde{Y}) \to H_1^{\delta^m}(X_\psi)$ is an epimorphism. Taking $m = d$, we have $H_1(\tilde{Y}, \mathbf{Z}_p) \to H_1(X_\psi, \mathbf{Z}_p)$ is onto. So the induced map $H_1(\tilde{Y}) \to H_1(X_\psi)/\text{torsion}$ when reduced mod p is onto. It follows that $H_1(\tilde{Y}, \mathbf{C}) \to H_1(X_\psi, \mathbf{C})$ is onto. Thus $\bar{\beta}_1(X_\psi) \leqslant \bar{\beta}_1(\tilde{Y}) = \rho_1(X) - 1$.

REMARK. To see why d is a power of p is a necessary hypothesis for (1.3) and (1.5), let X be the punctured torus bundle over S^1 with monodromy given by $h = [\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}]$. Since $\det(h - 1) = 1$, the Wang sequence shows X is a homology circle. In fact X is the complement of the trefoil knot. Since $h^6 = I$, the 6-fold cover $\tilde{X}$ is a trivial punctured torus bundle over S^1. In particular $\bar{\beta}_1(\tilde{X}) = 1 > \rho_1(X) - 1 = 0$ and $\beta_2(\tilde{X}) - \beta_2(X) = 2 > \rho_2(X) = 0$.

2. Codimension-0 embeddings. Throughout this section d is a power of p.

THEOREM 2.1. *Let M be a closed connected 4-manifold and N a codimension-0 submanifold with connected boundary L. For each primitive element z of the kernel of $H_2(N, \mathbf{Z}_d) \to H_2(M, \mathbf{Z}_d)$ there is some $\psi \in H^1(L, \mathbf{Z}_d)$ with $\delta(\psi)$ Lefschetz dual to z for which*

$$\begin{aligned} \rho_1(N) \geqslant & |\sigma(L, \psi) + \operatorname{sign} N - \operatorname{sign} M| - \rho_2(M) + \rho_2(N) + \rho_3(N) - 1 \\ & - \eta(L, \psi) + \max\begin{cases} \eta(L, \psi) - \eta(N) - \rho_1(M) + 1 \\ 0 \end{cases} \\ & + \max\begin{cases} \eta(L, \psi) - \eta(N) - \rho_3(N) + 1 \\ 0 \end{cases} \end{aligned}$$

where $\eta(N)$ is the nullity of $H_2(N, \mathbf{Z}_p) \to H_2(M, \mathbf{Z}_p)$.

PROOF. Let $X = M - \operatorname{Int} N$ so $\partial X = -L$ and consider the following commutative diagram with $\mathbf{Z}_d$ coefficients.

$$\begin{array}{ccccccc} H_3(M) & \to & H_3(M, N) & \to & H_2(N) & \to & H_2(M) \\ & & \uparrow \wr & & & & \\ H^1(X) & \underset{\text{LD}}{\overset{\approx}{\to}} & H_3(X, L) & \nearrow & & & \\ \downarrow & & \downarrow & & & & \\ H^1(L) & \underset{\text{LD}}{\overset{\approx}{\to}} & H_2(L) & & & & \end{array}$$

Pick $\psi' \in H^1(X, \mathbf{Z}_d)$ mapping to z. Since z is primitive, ψ' is primitive. Let $\psi \in H^1(L, \mathbf{Z}_d)$ be the restriction to L. The following commutative diagram with $\mathbf{Z}_d$ coefficients shows $\delta\psi$ is Lefschetz dual to z.

$$\begin{array}{ccc} H^1(L) & \to & H^2(N, \partial) \\ \|\mathrm{LD} & & \|\mathrm{LD} \\ H_2(L) & \to & H_2(N) \end{array}$$

The Mayer-Vietoris sequence shows X is connected. Since ψ' is primitive $X_{\psi'}$ (which we denote by $\tilde{X}$) is connected. $\tilde{X}$ may be used to calculate $\sigma(L, \psi)$. In fact

$$|\sigma_1(\tilde{X})| = |\sigma(L, \psi) + \operatorname{sign} N - \operatorname{sign} M|. \tag{1}$$

Using the Mayer-Vietoris sequence for $M = N \cup X$, Poincaré duality in M and Proposition 1.4 we have

$$\bar{\beta}_3(\tilde{X}) \leqslant \rho_3(X) \leqslant \rho_1(M) - \rho_3(N). \tag{2}$$

By considering the first diagram of this proof, only now with $\mathbf{Z}_p$ coefficients, using Poincaré duality in M and applying Proposition 1.5 we have

$$\bar{\beta}_1(\tilde{X}) \leqslant \rho_1(X) - 1 \leqslant \rho_1(M) + \eta(N) - 1. \tag{3}$$

Whenever we have an exact sequence of complex vector spaces with a $\mathbf{Z}_d$ action commuting with the maps, the exact sequence splits as a direct sum of eigenspace exact sequences. We can get a different bound on $\bar{\beta}_1(\tilde{X})$ using $\bar{H}_1(\tilde{L}) \to \bar{H}_1(\tilde{X}) \to \bar{H}_1(\tilde{X}, \tilde{L})$. By definition $\bar{\beta}_1(\tilde{L}) = \eta(L, \psi)$. Lefschetz duality and universal coefficients show $\bar{\beta}_1(\tilde{X}, \tilde{L}) = \bar{\beta}_3(\tilde{X})$. So we have

$$\bar{\beta}_1(\tilde{X}) \leqslant \eta(L, \psi) + \rho_1(M) - \rho_3(N). \tag{4}$$

Consider the sequence $\bar{H}_2(\tilde{X}) \xrightarrow{j} \bar{H}_2(\tilde{X}, \tilde{L}) \to \bar{H}_1(\tilde{L}) \to \bar{H}_1(\tilde{X})$. The map $H_2(\tilde{X}, \mathbf{C}) \to H_2(\tilde{X}, \tilde{L}, \mathbf{C})$ with respect to a suitable basis is given by a matrix that also represents the intersection pairing, see **[H-N-K**, p. 60]. Thus the map j above can be represented by a matrix that gives the hermitian form on $\bar{H}_2(\tilde{X})$. Let η be the nullity of this matrix. We have

$$\eta \geqslant \begin{cases} \eta(L, \psi) - \eta(N) - \rho_1(M) + 1 \\ 0 \end{cases} \tag{5}$$

and

$$\bar{\beta}_2(\tilde{X}) \geqslant |\sigma_1(\tilde{X})| + \eta. \tag{6}$$

By Proposition 1.1 $\bar{\chi}(\tilde{X}) = \chi(X)$. Since L is odd dimensional $\chi(L) = 0$ so $\chi(X) = \chi(M) - \chi(N)$. Since $\tilde{X}$ is connected, $\bar{\beta}_0(\tilde{X}) = 0$. We have

$$\bar{\beta}_2(\tilde{X}) = \chi(M) - \chi(N) + \bar{\beta}_1(\tilde{X}) + \bar{\beta}_3(\tilde{X}). \tag{7}$$

Putting together (1)–(7), writing out $\chi(M)$ and $\chi(N)$ in terms of mod p Betti numbers, using $\rho_1(M) = \rho_3(M)$, and simplifying leads directly to the result.

REMARK. If $\rho_1(M) = 0$ then by (2), $\rho_3(N) = 0$. Then the two max terms above will be equal. Since $x + |x| = \max(2x, 0)$ the conclusion of Theorem 2.1 can be rewritten in the case $\rho_1(M) = 0$ as

$$\rho_1(N) \geqslant |\sigma(L, \psi) + \operatorname{sign} N - \operatorname{sign} M| - \rho_2(M) + \rho_2(N)$$
$$- \eta(N) + |\eta(L, \psi) - \eta(N) + 1|.$$

A configuration of surfaces $\{F_i\}$ in a 4-manifold M is defined to be a map $\Sigma F_i \to M$ that arises in the following way. One starts with a smooth proper (boundary goes to boundary) embedding of $\Sigma \hat{F}_i \to M^4 - \bigcup \operatorname{Int} D_j^4$ where the D_j^4 are disjoint 4-balls in the interior of M and $\hat{F}_i$ are the surfaces F_i with a certain number of disjoint open 2-balls deleted. Thus each $S^3 = \partial D_j^4$ intersects the image of $\Sigma \hat{F}_i$ in a link $\mathfrak{L}_j$ and each component of $\mathfrak{L}_j$ "belongs" to a given F_i. By coning off each of these links to a central point in D_j^4, we get our map $\Sigma F_i \to M$. We refer to $\{\mathfrak{L}_j\}$ as the links of the configuration. By abuse of notation, we talk about configurations $\{F_i\}$ (thinking of each F_i as lying in M) and write $\bigcup F_i$ for the image and $[F_i] \in H_2[M]$ for the homology class represented by F_i. A neighborhood N of a configuration $\{F_i\}$ is defined to be the union in M of each D_j^4 and a closed tubular neighborhood of each $\hat{F}_i$. Define $\mu(\mathfrak{L})$ to be the number of components in a link $\mathfrak{L}$. We define the mod p nullity $\eta\{F_i\}$ of a configuration to be the number of surfaces less the dimension of the subspace spanned by $[F_i] \in H_2(M, \mathbf{Z}_p)$. We say a configuration is connected if $\bigcup F_i$ is.

COROLLARY 2.2. *Let $\{F_i\}$ be a connected configuration of n surfaces in a closed 4-manifold M with links $\mathfrak{L}_j$ and neighborhood N with boundary L. If $\Sigma a_i[F_i] = 0$ in $H_2(M, \mathbf{Z}_d)$ and $a_i \neq 0 \bmod p$ for some i, there is some $\psi \in H^1(L, \mathbf{Z}_d)$ with $\delta\psi$ Lefschetz dual to $\Sigma a_i[F_i] \in H_2(N, \mathbf{Z}_d)$ for which*

$$\begin{aligned}\sum \beta_1(F_i) \geqslant {} & |\sigma(L, \psi) + \operatorname{sign} N - \operatorname{sign} M| - \rho_2(M) + 2(n-1) \\ & - \eta(L, \psi) - \sum (\mu(\mathfrak{L}_j) - 1) \\ & + \max\begin{cases} \eta(L, \psi) - \eta\{F_i\} - \rho_1(M) + 1 \\ 0 \end{cases} \\ & + \max\begin{cases} \eta(L, \psi) - \eta\{F_i\} + 1 \\ 0. \end{cases}\end{aligned}$$

PROOF. N is homotopy equivalent to $\bigcup F_i$. By comparing the Mayer-Vietoris sequences for ΣF_i (regarded as the union of $\Sigma \hat{F}_i$ and some 2-disks) with the sequence for $\bigcup F_i$ (regarded as the union of $\Sigma \hat{F}_i$ and some cones on links) one sees that $\rho_2(\bigcup F_i) = n$. Clearly $\rho_3(\bigcup F_i) = 0$. A cell counting Euler characteristic argument then shows $\rho_1(\bigcup F_i) = \Sigma \beta_1(F_i) + \Sigma(\mu(\mathfrak{L}_j) - 1) - (n-1)$. These substitutions in (2.1) give the desired result.

THEOREM 2.3. *Let M be a closed $2k$-manifold and N a codimension-0 submanifold with boundary L. For each z in the kernel of $H_2(N, \mathbf{Z}_d) \to H_2(M, \mathbf{Z}_d)$, there is some $\psi \in H^1(L, \mathbf{Z}_d)$ with $\delta\psi$ Lefschetz dual to z for which*

$$\rho_{k-1}(N) \geqslant |\sigma(L, \psi) + \operatorname{sign} N - \operatorname{sign} M| - \rho_k(M).$$

PROOF. As in the proof of (2.1) let $X = M - \operatorname{Int} N$ and consider the following commutative diagram with $\mathbf{Z}_d$ coefficients.

$$\begin{array}{ccccccc} & & H_{2k-1}(M,N) & \to & H_{2k-2}(N) & \to & H_{2k-2}(M) \\ & & \uparrow\wr & & & & \\ H^1(X) & \overset{\approx}{\to} & H_{2k-1}(X,L) & \nearrow & & & \\ \downarrow & & \downarrow & & & & \\ H^1(L) & \to & H_{2k-2}(L) & & & & \end{array}$$

Again pick $\psi' \in H^1(X)$ mapping to z, and let ψ be the restriction of ψ' to $H^1(L)$, and $\tilde{X}$ denote $X_{\psi'}$. By Lefschetz duality, excision and the long exact sequence for the pair (M, N) we have

$$\rho_k(X) = \rho_k(X, L) = \rho_k(M, N) \leqslant \rho_k(M) + \rho_{k-1}(N).$$

By Proposition 1.4 $\bar{\beta}_k(\tilde{X}) \leqslant \rho_k(X)$. On the other hand

$$\beta_k(\tilde{X}) \geqslant |\sigma_1(\tilde{X})| = |\sigma(L, \psi) + \operatorname{sign} N - \operatorname{sign} M|.$$

The result follows easily.

REMARK. Theorem 2.1, and (2.2), (2.3) for $k \neq 1$, (4.1), (4.2) and (0.1) still hold if M has boundary a collection of $\mathbf{Z}_p$ homology spheres and N or $\bigcup F_i$ are in the interior of M. To see this let $\overline{M}$ be M union a cone on each boundary component. Then $\overline{M}$ is a $\mathbf{Z}_p$ homology manifold and Poincaré and Lefschetz duality still hold with $\mathbf{Q}$ or $\mathbf{Z}_p$ coefficients. Moreover since a $\mathbf{Z}_d$ cover restricted to the boundary must be trivial, it is easy to see that $\sigma(L, \psi)$ may still be calculated using $X = \overline{M} - \operatorname{Int} N$.

COROLLARY 2.4 (ITENBERG [$\mathbf{I}_2$]). *Let F be a codimension-2 submanifold of a closed $2k$-manifold M and suppose $[F] \in H_{2k-2}(M)$ is Poincaré dual to $x \in H^2(M)$ and $ax = dy$ where $y \in H^2(M)$ and $0 < a < d$. Then*

$$\rho_{k-1}(F) \geqslant \left|\{e^{2y}(1 - \tanh x)\mathfrak{L}(M)\}[M]\right| - \rho_k(M).$$

Here $\mathfrak{L}(M)$ denotes the total $\mathfrak{L}$ class of M as defined by Hirzebruch.

PROOF. Let N be a tubular neighborhood of F and $z = a[F] \in H_{2k-2}(N)$. In [$\mathbf{I}_1$, §6], Itenberg calculates using the G-Signature Theorem that

$$\sigma_a(\tilde{M}) = \{\exp((2a - d)x/d)\operatorname{sech}(x)\mathfrak{L}(M)\}[M]$$

for $\tilde{M}$ a d-fold branched cover of M along $[F]$. To complete the proof note that for any $\psi \in H^1(X, Z_d)$ with $\delta\psi$ Lefschetz dual to z, $\sigma(L, \psi) = \operatorname{Sign} M - \operatorname{Sign} N - \sigma_a(\tilde{M})$. We have also written Itenberg's power series in a slightly different form.

REMARK. The above proof of Itenberg's result is not substantially different from his own. It seems slightly easier to do the Smith Theory for unbranched covers. Theorem 2.3 is much more general. However one needs to be able to calculate $\sigma(L, \psi)$ in cases of interest. I have nearly proved the following conjecture. A strong deformation retract $F\colon A \times I \to A$ onto $B \subset A$ will be called very strong if $F^{-1}(B) = B \times I \cup A \times \{1\}$.

Conjecture 2.5. Let N^{2k} be a manifold with boundary L and $\{F_i\}$ a collection of n closed codimension-2 submanifolds Lefschetz dual to $x_i \in H^2(N, \partial)$. Suppose the F_i are in general position and that N very strongly deformation retracts onto $\bigcup F_i$. Given $\psi \in H^1(L, \mathbf{Z}_d)$ define integers a_i by $\delta\psi = \rho(\sum a_i x_i)$ where $0 \leqslant a_i < d$. If

$(a_i, d) = 1$ then

$$\sigma(L, \psi) + \operatorname{sign} N + \left\{ e^{2y} \prod_{i=1}^{n} (1 - \tanh x_i) \mathfrak{L}(N) \right\} [N, \partial] = 0$$

where $y = d^{-1} \sum a_i x_i \in H^2(N, \partial, \mathbf{Q})$.

That this is true for $n = 1$ will follow from Itenberg's formula. Proposition 5.2 proves the conjecture for $k = 1$. Theorem 3.7 and Proposition 3.8 will imply the conjecture for $k = 2$. However the proof of (2.5) when all the details are ironed out will be very different from the proof of (3.7). Finally we note

$$e^{2y} \prod (1 - \tanh x_i) = 1 + \left(2y - \sum x_i\right) + 2y\left(y - \sum x_i\right) + \sum_{i<j} x_i x_j + \text{terms of higher degree.}$$

This conjecture is made without the requirement that d be a power of p.

3. Invariants of links and finite cyclic covers of 3-manifolds. By a link $\mathfrak{L}$, we will mean a collection of oriented disjoint smoothly embedded circles K_i in S^3. A represented link $(\mathfrak{L}, \psi)$ is a link together with a map ψ: $H_1(S^3 - \mathfrak{L}) \to \mathbf{Z}_d$. We orient the meridian m_i of each K_i so m_i and K_i link positively. $H^1(S^3 - \mathfrak{L})$ is a free abelian group generated by $\{m_i\}$. Thus ψ may be described by the sequence of mod d integers $\psi(m_i)$. Define the nullity $\eta(\mathfrak{L}, \psi)$ of a represented link to be $\bar{\beta}_1((S^3 - \mathfrak{L})_\psi)$. We will refer to the link pictured in Figure 1 as the positive Hopf link. Reversing the orientation on one component gives the negative Hopf link.

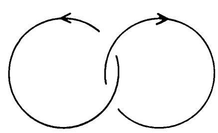


FIGURE 1

PROPOSITION 3.1. *If d is a prime power, $\eta(\mathfrak{L}, \psi) \leqslant \mu(\mathfrak{L}) - 1$. If $\mathfrak{L}$ is a Hopf link and ψ is an epimorphism, then $\eta(\mathfrak{L}, \psi) = 0$.*

PROOF. The first statement follows immediately from (1.5). The complement of the Hopf link deformation retracts to a torus. A connected cover of a torus is a torus homologically fixed by the covering translation.

A represented link $(\mathfrak{L}, \psi)$ is called well represented if for each i, $\psi(m_i)$ is a generator for $\mathbf{Z}_d$.

Any closed 3-manifold L arises by doing surgery on a framed link $\mathfrak{L}$ with framings, say n_i. Let n_{ij} for $i \neq j$ be the linking number of K_i and K_j and $n_{ii} = n_i$. If ψ: $H_1(L) \to \mathbf{Z}_d$, then ψ is determined by $\psi(m_i)$. Moreover specifying $\psi(m_i)$ determines a map ψ: $H_1(S^3 - \mathfrak{L}) \to \mathbf{Z}_d$ which extends over all of $H_1(L)$ if and only if $\sum_j n_{ij}\psi(m_j) = 0 \bmod d$ for all i. In this case we say the n_i form a compatible framing for $(\mathfrak{L}, \psi)$. If $(\mathfrak{L}, \psi)$ is well represented, there exist compatible framings (which are determined mod d).

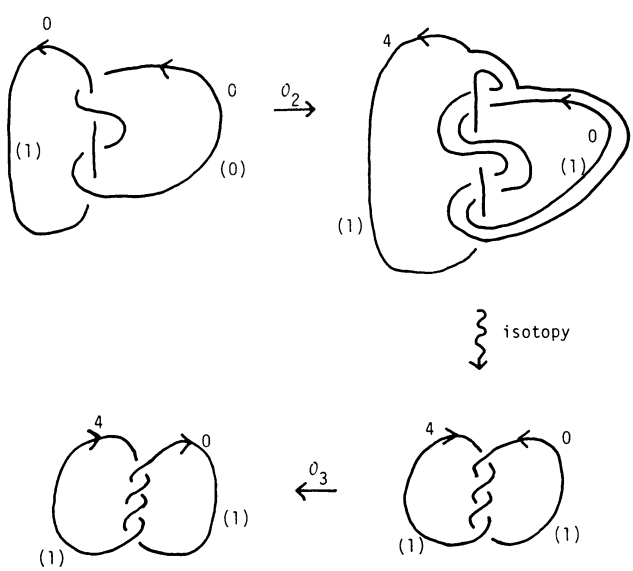
PROPOSITION 3.2. *Let (L, ψ) be given by placing a compatible framing on a well represented link $(\mathfrak{L}, \psi)$. Let S^3_ψ denote the branched cover of S^3 along $\mathfrak{L}$ given by ψ. Then $\eta(\mathfrak{L}, \psi) = \bar{\beta}_1(S^3_\psi) = \eta(L, \psi)$.*

PROOF. The eigenspace Mayer-Vietoris sequence for S^3_ψ as the union of $(S^3 - \mathfrak{L})_\psi$ and solid tori along their boundaries shows $\eta(\mathfrak{L}, \psi) = \bar{\beta}_1(S^3_\psi)$ as both the solid tori and their boundaries are homologically fixed by $\mathbf{Z}_d$. The same argument works for L_ψ.

Equivalently we can think of L as the boundary of N, the 4-manifold obtained by attaching 2-handles to D^4 along $\mathfrak{L}$ with the attaching map specified by the framings n_i (see [**K**]). $H_2(N)$ is free abelian with naturally given basis x_i formed by the cores of the 2-handles union the cones on K_i in D^4. The intersection form relative to this basis is given by the matrix $[n_{ij}]$ and the compatibility condition says that $\Sigma\, \psi(m_i)x_i$ is in the kernel of $H_2(N, \mathbf{Z}_d) \to H_2(N, \partial, \mathbf{Z}_d)$.

Kirby [**K**] has defined certain moves on framed links which allow one to change from one framed link picture of L to any other. Given a framed link picture of L, one can display ψ by placing $\psi(m_i)$ in parentheses near K_i. As one makes moves in the Kirby calculus, one can keep track of ψ as follows. In move $\mathcal{O}_1$, one places (0) near the added unknotted component. In move $\mathcal{O}_2$, where one "adds" K_i to K_j (here we insist that the band used in forming the sum be compatible with the orientations of both components) the value of $\psi(m_j)$ remains the same but the new $\psi(m_i)$ is $\psi(m_i) - \psi(m_j)$ in terms of the original values. Finally since we are working with oriented links, we need a third move $\mathcal{O}_3$ which allows one to change the orientation of a component K_i and simultaneously the value $\psi(m_i)$ to $-\psi(m_i)$ mod d.

EXAMPLE 3.3. $d = 2$.



We now develop a procedure to calculate $\sigma(L, \psi)$ and $\eta(L, \psi)$, given the above type link descriptions of (L, ψ) in terms of corresponding link invariants. Given a number $0 < q < 1$ and a square complex valued matrix V define

$$V_q = (1 - e^{2\pi i q})V + (1 - e^{-2\pi i q})V^*.$$

Define the q-signature and q-nullity of a link $\mathfrak{L}$ to be $\sigma_q(\mathfrak{L}) = \sigma(V_q)$ and $\eta_q(\mathfrak{L}) = \eta(V_q)$ where V is a Seifert matrix belonging to a connected Siefert surface for $\mathfrak{L}$.

This is a slightly different notation for the signatures defined by J. Levine **[Le]**. The nullity defined above is smaller by one than the usual one **[T]**, **[K-T]**, **[Ka$_2$]**.

PROPOSITION 3.4. *Let F be a connected Seifert surface for a link $\mathfrak{L}$ in S^3 with Seifert matrix V. Let $\tilde{D}^4$ be the d-fold branched cyclic cover of D^4 along F pushed into D^4. Then the intersection form on $H_2(\tilde{D}^4, s)$ can be given by the matrix $V_{(s/d)}$. Moreover $H_1(\tilde{D}^4) = 0$.*

PROOF. We first remark that the consequence $\sigma_{(s/d)}(\mathfrak{L}) = \sigma_s(\tilde{D}^4)$ is originally due to Viro **[V$_1$]**. (3.4) for $d = 2$ appears in **[K-T]**. That $H_1(\tilde{D}^4) = 0$ follows from Kauffman's cut and paste description of $\tilde{D}^4$ **[Ka$_1$]**. The rest of this proposition follows from this description and a little algebra. See **[D-K**, Theorem 5.1].

REMARK. It follows that $\eta_{(s/d)}(\mathfrak{L}) = \bar{\beta}_1(\partial\tilde{D}^4)$. Using **[T-W**, Lemma 7.2], $\eta_{(s/d)}(\mathfrak{L}) = \eta_{(s'/d)}(\mathfrak{L})$ if $(s, d) = (s', d) = 1$.

PROPOSITION 3.5. *Let L be the boundary of a 4-manifold W and $\psi \in H^1(L, \mathbf{Z}_d)$. Suppose $\delta\psi$ is Lefschetz dual to $\rho(\sum a_i[F_i])$ where F_i is a collection of disjoint, smoothly embedded surfaces in W, and $(a_i, d) = 1$. There exists*

$$\psi' \in H^1\left(W - \bigcup F_i\right)$$

extending ψ taking the value a_i on a positive meridian of F_i. Let $\tilde{W}$ be the associated branched cover of W. Then we have for $0 < s < d$

$$\sigma(L, s\psi) = \sigma_s(\tilde{W}) - \operatorname{sign} W + \frac{2}{d^2}\sum (d - b_i)b_i(F_i \circ F_i)$$

where $b_i = sa_i \bmod d$ and $0 < b_i < d$.

PROOF. Consider the following commutative diagram with $\mathbf{Z}_d$ coefficients:

$$\begin{array}{ccccccc}
H^1(W) & \to & H^1(L) & \to & H^2(W, L) & \to & H^2(W) \\
\uparrow\| & & \uparrow & & \uparrow j & & \uparrow\| \\
H^1(W) & \to & H^1(W - \cup F_i) & \to & H^2(W, W - \cup F_i) & \to & H^2(W)
\end{array}$$

The Thom isomorphism and excision give an isomorphism Φ: $H^1(\cup F_i, \mathbf{Z}_d) = H^2(W, W - \cup F_i, \mathbf{Z}_d)$. Let e_i be the generator of $H^0(F_i, \mathbf{Z}_d)$ (Poincaré dual to the oriented fundamental class). Then $j\Phi(\sum a_i e_i) = \delta\psi$ (as both sides are Lefschetz dual to $\rho(\sum a_i[F_i])$). An easy diagram chase then manufactures $\psi' \in H^1(W - \cup F_i, \mathbf{Z}_d)$ with the stated properties.

Let L_i be the boundary of N_i, the tubular neighborhood of F_i. By (5.4) (using $d/(d, s)$ in place of d) we have

$$\sigma(L_i, s\psi') = 2(d - b_i)b_i(F_i \circ F_i)/d^2 - \operatorname{sign} N_i.$$

This may also be deduced from Rokhlin's formula (18) **[R]**. Let $V = W - \cup \operatorname{Int} N_i$ and $\tilde{V}$ the cover given by ψ'. A Mayer-Vietoris sequence shows $\sigma_s(\tilde{V}) = \sigma_s(\tilde{W})$. Novikov additivity shows $\operatorname{sign} V = \operatorname{sign} W - \sum \operatorname{sign} N_i$. Finally

$$\sigma(L, s\psi) - \sum \sigma(L_i, s\psi) = \sigma_s(\tilde{V}) - \operatorname{sign} V.$$

This completes the proof.

If K is a component of a link, an algebraic r-cable with twist n along K is obtained by pushing off algebraically r copies of K with framing n with respect to the null-homologous framing from K. A cable is called nonempty if at least one copy of K is pushed off.

THEOREM 3.6. *Let (L, ψ) be given by a represented link $(\mathfrak{L}, \psi)$ together with a compatible framing $\{n_i\}$. Assume ψ is an epimorphism. Let n_{ij} be defined as above. For any integers r_i with $r_i = \psi(m_i)$ mod d, let $\mathfrak{L}'$ be obtained from $\mathfrak{L}$ by replacing each K_i with a nonempty algebraic r_i-cable with twist n_i along K_i. Then for $0 < s < d$*

$$\sigma(L, s\psi) = \sigma_{(s/d)}(\mathfrak{L}') - \operatorname{sign}[n_{ij}] + \frac{2(d-s)s}{d^2} \sum r_i r_j n_{ij},$$

$$\eta(L, s\psi) = \eta_{(s/d)}(\mathfrak{L}') - \mu(\mathfrak{L}') + \mu(\mathfrak{L}).$$

REMARKS. (a) The formula for $\sigma(L, s\psi)$ above when $r_i = 1$ and $\mathfrak{L}' = \mathfrak{L}$ is due to Casson and Gordon [**C-G$_2$**, Lemma 3.1] although I was unaware of this when I derived (3.6). As pointed out in [**C-G$_2$**], one can calculate $\sigma(L, s\psi)$, given any framed link for (L, ψ) by first doing moves in the calculus of framed links until the r_i can all be chosen to be one and then apply their special case.

(b) The nonempty condition is necessary. To see this consider (L, ψ) of Example 3.3. If we could drop the nonempty condition, then using the first description one would calculate $\sigma(L, \psi) = 0$. On the other hand, using the final description, one sees $\sigma(L, \psi) = 1$, in fact. However, we can use the first description to calculate $\sigma(L, \psi) = \sigma_{(1/2)}(\mathfrak{L}') = 1$ where $\mathfrak{L}'$ is obtained by replacing the right-hand component with a nonempty algebraic 0-cable with twist zero. See Figure 2.

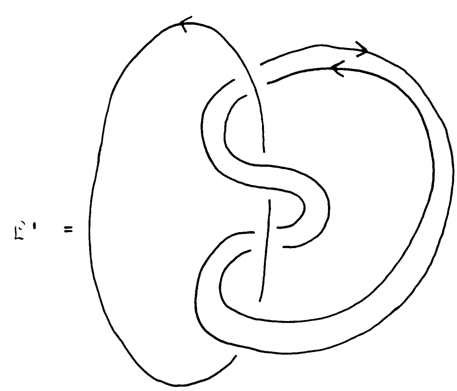


FIGURE 2

(c) We can use (3.6) to see that the well represented hypothesis is necessary in (3.2). Consider the represented link given by the first picture in Example 3.3. It is easy to see $\eta(\mathfrak{L}, \psi) = 1$. On the other hand, using the last description and (3.6), $\eta(L, \psi) = 0$.

(d) Given a link $\mathfrak{L}$, we can make it well represented by defining $\psi(m_i) = 1$ for all i. Let n_i be a compatible framing (n_i are determined mod d) and let $\mathfrak{L}'$ be formed from $\mathfrak{L}$ as in (3.6). By identifying the two expressions one gets for $\sigma(L, s\psi)$ one obtains a relation between $\sigma_{(s/d)}(\mathfrak{L})$ and $\sigma_{(s/d)}(\mathfrak{L}')$. Similarly one gets a relation between $\eta_{(s/d)}(\mathfrak{L})$ and $\eta_{(s/d)}(\mathfrak{L}')$.

In particular let $l = \sum_{j\neq 1} n_{1j}$ and $\mathfrak{L}'$ be formed by replacing K_1 by a nonempty algebraic $dq + 1$ cable with twist f and $\tau + 1$ components in $\mathfrak{L}$ where $f + l = 0$ mod d. We then have

$$\sigma_{(s/d)}(\mathfrak{L}') = \sigma_{(s/d)}(\mathfrak{L}) - \frac{2(n-s)s}{d^2}\left[fd^2q^2 + 2dq(f+l)\right]$$

and

$$\eta_{(s/d)}(\mathfrak{L}') = \eta_{(s/d)}(\mathfrak{L}) + \tau.$$

If we specialize to $d = p$, $s = [p/2]$, $q = 0$ and $\tau = 2$, this yields Tristram's Theorem 3.2 [T].

On the other hand, if $d = 2$ and letting $\mathfrak{L}'$ be $\mathfrak{L}$ with the orientation on some of the components changed one sees $\sigma(\mathfrak{L}) + \sum_{i<j} n_{ij}$ is invariant of the orientation of the components of $\mathfrak{L}$. This is a theorem of Murasugi **[M$_2$]**. See also **[K-T]** and **[G-L]** for other proofs.

Proof of Theorem 3.6. Let F be a pushed in Seifert surface for $\mathfrak{L}'$. Let N be D^4 with 2-handles H_i attached along $\mathfrak{L}$ with framing n_i so that $L = \partial N$. Let F' be the closed embedded surface formed by F and algebraically r_i push offs of the core of H_i. Thus $F' \cap S^3 = \mathfrak{L}'$. It is easy to see that $F' \circ F' = \sum r_i r_j n_{ij}$ and that $\delta(\psi)$ is Lefschetz dual to $\rho[F']$. By (3.5) we can take the d-fold branched cover $\tilde{N}$ of N along F' and

$$\sigma(L, s\psi) = \sigma_s(\tilde{N}) - \operatorname{sign}[n_{ij}] + \frac{2(d-s)s}{d^2}\sum r_i r_j n_{ij}.$$

Let D_i be a 2-disk in S^3 transverse to the cable along K_i. The cable will intersect D_i in a finite set of (signed) points which when counted algebraically sum to r_i, but counted geometrically sum to, say, $\tau_i + 1$. $\tilde{D}_i$, the cover restricted to D_i, is a connected surface and $\beta_1(\tilde{D}_i) = (d-1)\tau_i$. By considering the collection of quotient covers and using the Thomas-Wood argument $\beta_1(\tilde{D}_i, s) = \tau_i$.

Let $\tilde{H}_i$ and $\tilde{D}$ be the covers restricted to H_i and D_4 and $\tilde{H}_i$ be attached to $\tilde{D}$ along A_i. Let $\hat{H}_k(\ )$ denote $H_k(\ , s)$. We have $\tilde{H}_i = \tilde{D}_i \times D^2$ and $A_i = \tilde{D}_i \times S^1$. $\hat{H}_2(\tilde{H}_i) = 0$, $\hat{H}_0(A_i) = 0$ and by (3.4) $\hat{H}_1(\tilde{D}) = 0$. $H_1(A_i) \to H_1(\tilde{H}_i)$ is surjective with kernel infinite cyclic and fixed by $\mathbf{Z}_d$. It follows that $\hat{H}_1(A_i) \to \hat{H}_1(\tilde{H}_i)$ is an isomorphism.

These facts together with the following Mayer-Vietoris sequence show $\hat{H}_1(\tilde{N}) = 0$.

$$\bigoplus_i \hat{H}_1(A_i) \to \hat{H}_1(\tilde{D}_4) \bigoplus_i \hat{H}_1(\tilde{H}_i) \to \hat{H}_1(\tilde{N}) \to \bigoplus_i \hat{H}_0(A_i).$$

Since $\hat{H}_1(\tilde{N}) = 0$, Lefschetz duality and universal coefficients show $\hat{H}_3(\tilde{N}, L_\psi) = 0$. The long exact sequence

$$H_4(\tilde{N}, L_\psi) \overset{\approx}{\to} H_3(L_\psi) \to H_3(\tilde{N}) \to H_3(\tilde{N}, L_\psi)$$

then shows $\hat{H}_3(\tilde{N}) = 0$.

A different part of the same Mayer-Vietoris sequence then gives

$$0 \to \bigoplus \hat{H}_2(A_i) \to \hat{H}_2(\tilde{D}) \to \hat{H}_2(\tilde{N}) \to 0.$$

It is easy to see $\dim \hat{H}_2(A_i) = \tau_i$. Moreover the image of $H_2(A_i)$ in $H_2(\tilde{D})$ is annihilated by the form. It follows $\sigma_s(\tilde{N}) = \sigma_s(\tilde{D})$ and the nullity of the intersection form on $\tilde{N}$ plus $\Sigma\, \tau_i$ is the nullity of the intersection form on $\tilde{D}$. The long exact sequence

$$\hat{H}_2(\tilde{N}) \to \hat{H}_2(\tilde{N}, L_\psi) \to \hat{H}_1(L_\psi) \to \hat{H}_1(\tilde{N}) = 0$$

shows $\eta(L, s\psi)$ equals the nullity of the intersection form on $\tilde{N}$. Using $\mu(\mathfrak{L}') - \mu(\mathfrak{L}) = \Sigma\, \tau_i$ and Proposition 3.4 together with the above facts, the result follows.

Using the above methods it is now possible to calculate $\sigma(L, \psi)$ and $\eta(L, \psi)$ given a link description of (L, ψ). Moreover for L arising as the boundary of a configuration it is possible (in most cases) to give formulas for these invariants in terms of the corresponding invariants of the links that describe the singularities. The rest of this section is concerned with this.

Let $(\mathfrak{L}, \psi)$ be a well represented link and $n_i = n_{ii}$ some compatible framing. Let $r_i = \psi(m_i)$ where $0 < r_i < d$ and define vectors $r = (r_1, \ldots, r_k)$, $\bar{r} = (d - r_1, \ldots, d - r_k)$ and let $\langle r, \bar{r}\rangle$ denote $r[n_{ij}]\bar{r}^T$. Let (L, ψ) be given by $(\mathfrak{L}, \psi)$ and n_i. Define

$$\sigma(\mathfrak{L}, \psi) = \sigma(L, \psi) + \operatorname{sign}[n_{ij}] - \frac{2}{d^2}\langle r, \bar{r}\rangle.$$

$\sigma(\mathfrak{L}, \psi)$ is independent of the choice of n_i and is thus a well represented link invariant. To see this, let n_i and n_i' be two choices. Attach handles with framing n_i to $S^3 \times I$ along $\mathfrak{L} \subset S^3 \times \{0\}$ and framing $-n_i'$ along $-\mathfrak{L} \subset -S^3 = S^3 \times \{1\}$ to form N. The cores of the handles union $\mathfrak{L} \times I$ form surfaces $F_i \subset N$ and $F_i \circ F_i = n_i - n_i'$. An application of (3.5) and a Mayer-Vietoris sequence argument complete the proof. For more details see the proof of Theorem 3.7 below which is a more complicated version of this argument.

One may use (3.6) to calculate $\sigma(\mathfrak{L}, \psi)$. In fact let $0 < s < d$, $(s, d) = 1$, $r_i = \psi(m_i) \bmod d$, $q_i = sr_i \bmod d$, $0 < q_i < d$, $q = (q_1, \ldots, q_k)$ and $\bar{q} = (d - q_1, \ldots, d - q_k)$. We have

$$\sigma(\mathfrak{L}, s\psi) = \sigma_{(s/d)}(\mathfrak{L}') + \frac{2}{d^2}((d - s)s\langle r, r\rangle - \langle q, \bar{q}\rangle)$$

where $\mathfrak{L}'$ is obtained from $\mathfrak{L}$ by replacing each K_i with a nonempty algebraic r_i-cable with twist n_i. Here $n_i = n_{ii}$ is a compatible framing for $(\mathfrak{L}, \psi)$ and $\langle x, y\rangle$ indicates $x[n_{ij}]y^T$. In particular if $\psi(m_i) = 1$ for all i, then $\sigma(\mathfrak{L}, s\psi) = \sigma_{(s/d)}(\mathfrak{L})$. Thus $\sigma(\mathfrak{L}, \psi)$ generalizes Levine's signatures. Let $(-\mathfrak{L}, \psi)$ denote the well represented link obtained by changing all the crossings of $(\mathfrak{L}, \psi)$. It is not hard to see $\sigma(-\mathfrak{L}, \psi) = -\sigma(\mathfrak{L}, \psi)$.

We remind the reader that for a well represented link as above, by (3.2) and (3.6)

$$\eta(\mathfrak{L}, s\psi) = \eta_{(s/d)}(\mathfrak{L}') + \mu(\mathfrak{L}) - \mu(\mathfrak{L}').$$

THEOREM 3.7. *Let F_i be a collection of closed surfaces and $\hat{F}_i$ be these surfaces with the interiors of some disjoint 2-disks removed. Let N be formed by attaching $\hat{F}_i \times D^2$ to ΣD_j^4 by identifying $\Sigma\, \partial \hat{F}_i \times D^2$ with tubular neighborhoods of some links $\mathfrak{L}_j$ in ∂D_j^4. By coning off the links we get a configuration of surfaces $\{F_i\}$ in N.*

Let $L = \partial N$ and $\psi \in H^1(L, \mathbf{Z}_d)$ be such that $\delta\psi$ is Lefschetz dual to $\rho(\Sigma\, a_i[F_i])$ where $(a_i, d) = 1$ and $0 < a_i < d$. By assigning to the meridian of a component K of $\mathfrak{L}_j$ the value a_i if K belongs to F_i, we obtain some well represented links $(\mathfrak{L}_j, \psi)$. Let $z = \Sigma\, a_i[F_i]$ and $\bar{z} = \Sigma(d - a_i)[F_i]$, then we have

$$\sigma(L, \psi) = \sum \sigma(\mathfrak{L}_j, \psi) + \frac{2}{d^2}(z\bar{z}) - \operatorname{sign} N,$$

and

$$\eta(L, \psi) = \sum \eta(\mathfrak{L}_j, \psi)$$

where $z\bar{z}$ stands for the intersection of z and $\bar{z}$ in $H_2(N)$.

PROOF. Let D_j' be a smaller concentric 4-disk in each D_j^4. Construct N' as follows. Remove Int D_j' from each D_j^4 and then attach 2-handles along $\mathfrak{L}_j$ with framings compatible to $(\mathfrak{L}_j, \psi)$ to what remains. This is pictured schematically in Figure 3. We have $L' = \partial N' = L - \Sigma\, L_j$ where L_j is gotten by doing surgery to S^3 according to $\mathfrak{L}_j$ so framed. Each L_j inherits $\psi \in H^1(L_j, \mathbf{Z}_d)$ from $(\mathfrak{L}_j, \psi)$. Using the cores of the new 2-handles we have a natural embedding of $\Sigma\, F_i'$ in N' where F_i' is another copy of F_i. Define $\psi' \in H^1(L', \mathbf{Z}_d)$ using (L, ψ) and (L_j, ψ). We have

$$\sigma(L', \psi) = \sigma(L, \psi) - \sum \sigma(L_j, \psi). \tag{1}$$

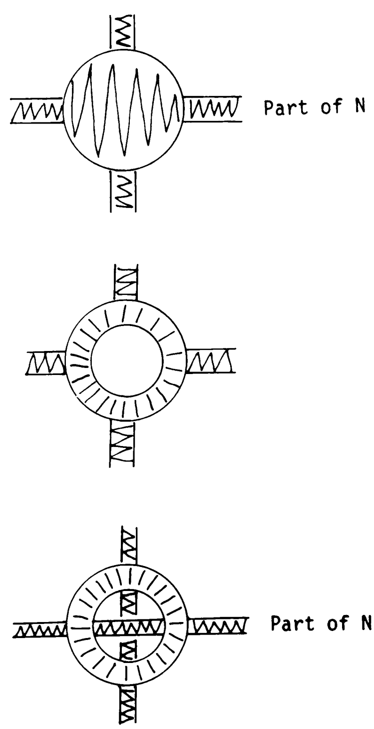


FIGURE 3

Let N_j be formed by attaching 2-handles to S^3 along $-\mathfrak{L}_j$ with the negatives of the above framings. $H_2(N_j)$ is free abelian generated by classes x^j_α represented by a pushed in Seifert surface for K^j_α (the αth component of $-\mathfrak{L}_j$) union the core of appropriate handle. Let $r^j = \Sigma\, a^j_\alpha x^j_\alpha$ and $\bar{r}^j = \Sigma(d - a^j_\alpha)x^j_\alpha$ where $a^j_\alpha = a_i$ if K^j_α belongs to F_i. Since $L_j = -\partial N_j$,

$$\sigma(\mathfrak{L}_j, \psi) = \sigma(L_j, \psi) - \operatorname{sign} N_j + 2(r^j\bar{r}^j)/d^2. \tag{2}$$

If we view N' as $N \#_j N_j$ and $H_2(N')$ as $H_2(N) \oplus_j H_2(N_j)$, then $[F_i'] = [F_i] + \Sigma_{\Gamma_i} x^j_\alpha$ where $(j, \alpha) \in \Gamma_i$ if K^j_α belongs to F_i. One can see that $\delta\psi'$ is Lefschetz dual to $\rho(\Sigma\, a_i[F_i'])$. So we can use (3.5) to calculate $\sigma(L', \psi')$. The above decomposition of $H_2(N')$ is orthogonal. Moreover $F_i' \circ F_j' = 0$ if $i \neq j$. So we have

$$\begin{aligned}\sum (d - a_i)a_iF_i' \circ F_i' &= \left(\sum a_i[F_i']\right)\left(\sum (d - a_i)[F_i']\right)\\ &= \left(\sum_i a_i\left([F_i] + \sum_{\Gamma_i} x^j_\alpha\right)\right)\left(\sum_i (d - a_i)\left([F_i] + \sum_{\Gamma_i} x^j_\alpha\right)\right)\\ &= z\bar{z} + \sum_j r^j\bar{r}^j.\end{aligned}$$

Let $\tilde{N}'$ be the branched cover of N' along $\cup F'$ given by (3.5). We have

$$\sigma(L', \psi') = \sigma_1(\tilde{N}') - \operatorname{sign} N' + 2\left(z\bar{z} + \sum r^j\bar{r}^j\right)/d^2 \tag{3}$$

and

$$\operatorname{sign} N' = \operatorname{sign} N + \sum \operatorname{sign} N_j. \tag{4}$$

If one thinks of N' as the union of disk bundles over F' and $(S_j^3 - \mathfrak{L}_j) \times I$ for each j and decomposes the cover $\tilde{N}'$ accordingly, our usual Mayer-Vietoris argument shows that the intersection form on $\overline{H}_2(\tilde{N}')$ is identically zero. Thus $\sigma_1(\tilde{N}') = 0$. Together with (1)–(4) this gives the stated formula for $\sigma(L, \psi)$.

Since $(\mathfrak{L}_j, \psi_j)$ is well represented $\partial(\hat{F}_i \times S^1)_\psi$ is homologically fixed by $\mathbf{Z}_d$. $(\hat{F}_i \times S^1)_\psi = \hat{F}_i \times S^1$ is also homologically fixed. Thus the Mayer-Vietoris sequence for L_ψ as the union of $(\hat{F}_i \times S^1)_\psi$ and $(S^3 - \mathfrak{L}_j)_\psi$ gives the final formula.

PROPOSITION 3.8. *If $(\mathfrak{L}, \psi)$ is any well represented positive Hopf link $\sigma(\mathfrak{L}, \psi) = -1$.*

PROOF. Let $r_i = \psi(m_i) \bmod d$ where $0 < r_i < d$. Pick compatible n_i with $n_i < -1$. Then $\operatorname{sign}[n_{ij}] = -2$. Let $q_i = -n_i$. Then mod d, $r_1 = q_2r_2$ and $r_2 = q_1r_1$. (L, ψ) is the boundary of a plumbing and

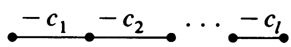
$$(L, \psi) = (L(q_1q_2 - 1, q_2), q_2r_2\chi)$$

where χ is the map specified in [**C-G$_1$**, pp. 6–7]. So $\sigma(L, \psi)$ is 4(area Δ − Int Δ) where Δ is the right triangle in the plane with vertices (0, 0), $((q_1q_2 - 1)r_2/d, 0)$ and $((q_1q_2 - 1)r_2/d, r_2q_2/d)$. See Example 3.9 below for more details and the definition of Int Δ. The line $x = q_1y$ goes through an integral lattice point for each integral y. The hypotenuse lies on the line $x = (q_1 - (1/q_2))y$. It is easy to see that all integral points in Δ lie on or to the right of the line $x = q_1y$. Also $[r_2q_2/d] = (r_2q_2 - r_1)/d$. Under these circumstances Int Δ and thus $\sigma(\mathfrak{L}, \psi)$ is an elementary if tedious calculation.

REMARKS. (a) If $\psi(m_1) = \psi(m_2)$, then $\sigma(\mathfrak{L}, \psi) = \sigma_{(s/d)}(\mathfrak{L}) = -1$. Similarly I have checked (3.8) in the case $\psi(m_1) = 2\psi(m_2)$ using the formula involving $\sigma_{s/d}(\mathfrak{L}')$, however this is considerably more difficult.

(b) Let N be a plumbing of 2-disk bundles over l surfaces according to a weighted graph with k edges and matrix B. Suppose $\psi \in H^1(\partial N, \mathbf{Z}_d)$ and let a_i $(0 < a_i < d)$ be defined to be the value ψ assigns to the circle fiber over a point in the ith surface. Let $a = (a_1, \ldots, a_l)$ and $\bar{a} = (d - a_1, \ldots, d - a_l)$. If $(a_i, d) = 1$ for all i, then $\sigma(\partial N, \psi) = 2(aB\bar{a}^T)/d^2 - \operatorname{sign} B - k$. This follows from (3.7) and (3.8). It also follows from Conjecture 2.5.

EXAMPLE 3.9. According to Hirzebruch **[H-N-K**, p. 70**]**, the *lens space* $L(n, q)$ is the boundary of the plumbing



where

$$n/q = [c_1, \ldots, c_l] = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{\ddots}{\ddots - \cfrac{1}{c_l}}}}.$$

Let $d|n$ and χ be the element of $H^1(L(n, q), \mathbf{Z}_d)$ which assigns to a circle fiber over a point on the first 2-sphere (with weight $-c_1$) the value 1. One can check that this is the same χ as specified by Casson and Gordon on pp. 6–7 of **[C-G$_1$]**. They show for $0 < r < d$

$$\sigma(L(n, q), rq\chi) = 4(\text{area } \Delta - \text{Int } \Delta)$$

where Δ is the triangle with vertices (0, 0), $(nr/d, 0)$, $(nr/d, qr/d)$. Int Δ is the number of integer lattice points in Δ, where boundary points count $1/2$, vertices count $1/4$, and (0, 0) is not counted.

On the other hand if we write $n/q = [c_1, \ldots, c_n]$ where $c_i > 1$ and given $0 < a < d$ define a_i recursively by $a_0 = 0$, $a_1 = a$ and $a_{i+1} = c_i a_i - a_{i-1} \bmod d$ and $0 < a_{i+1} < d$, then by Remark (b) above

$$\sigma(L(n, q), a\chi) = 1 - \frac{2}{d^2}\left(\sum_{i=1}^{l} (d - a_i)a_i c_i + \sum_{i=1}^{l-1} (2a_i a_{i+1} - d(a_i + a_{i+1}))\right).$$

If $(a, d) \neq 1$, then interpret $a\chi$ as a map $H_1(L) \to \mathbf{Z}_{d'}$ where $d' = d/(a, d)$. To apply Remark (b) in this situation, we also need $(a, d) = (a_i, d)$ for all i. However it is possible to reduce to this case by "blowing up and down" along the graph. It turns out the above formula holds as stated.

Casson was already aware of this formula or one like it. According to Casson, the equivalence of these two formulas (choose a in the second to equal rq mod d) is due to Eisenstein although I have not been able to find the reference.

4. Main results. Throughout this section d will be a power of p.

THEOREM 4.1. *Let $\{F_i\}$ be a connected configuration of n surfaces with links $\mathfrak{L}_j$ in a closed 4-manifold M. Let $x_i = [F_i] \in H_2(M)$ and $z = \sum a_i x_i = dy$ where $0 < a_i < d$ and $a_i \neq 0 \bmod p$. Use the a_i to make the $\mathfrak{L}_j$ well represented links $(\mathfrak{L}_j, \psi)$. We then have*

$$\sum \beta_1(F_i) + \sum \left(\mu(\mathfrak{L}_j) - 1 \right) \geqslant \left| 2y\left(\sum x_i - y \right) + \sum \sigma(\mathfrak{L}_j, \psi) - \operatorname{sign} M \right|$$
$$- \rho_2(M) + 2(n-1) - \sum \eta(\mathfrak{L}_j, \psi)$$
$$+ \max \begin{cases} \sum \eta(\mathfrak{L}_j, \psi) - \eta\{F_i\} - \rho_1(M) + 1 \\ 0 \end{cases}$$
$$+ \max \begin{cases} \sum \eta(\mathfrak{L}_j, \psi) - \eta\{F_i\} + 1 \\ 0. \end{cases}$$

PROOF. Apply (2.2) and then use (3.7) to evaluate $\sigma(L, \psi) + \operatorname{sign} N$ and $\eta(L, \psi)$.

If an F_i in a configuration has an intersection with itself that is given by a Hopf link this intersection is called an ordinary double point (positive or negative accordingly as the Hopf link is). The algebraic number of double points of a configuration is the sum over these points of their indices ± 1.

COROLLARY 4.2. *Let $\{F_i\}$ and a_i be as above. Assume in addition that each $\mathfrak{L}_j$ is either a Hopf link or a knot. Let # be the total number of Hopf links. Let $\{K_i^l\}_{l=1}^{q_i}$ be the collection of knots belonging to F_i and I the algebraic number of double points. We have*

$$\# + \sum \beta_1(F_i) \geqslant \left| 2y\left(\sum x_i - y \right) - \sum_{i<j} x_i x_j - I + \sum_i \sum_l \sigma_{(a_i/d)}(K_i^l) - \operatorname{sign} M \right|$$
$$- \rho_2(M) + 2(n-1).$$

PROOF. The nullity of a well represented knot or Hopf link is zero (3.1). The signature of a well represented Hopf link is ± 1 accordingly. Since $z = dy$, $\eta\{F_i\} > 0$. The algebraic number of Hopf links that do not give ordinary double points is $\sum_{i<j} x_{ij}$. The formula for the signature of a well represented link completes the proof.

REMARKS. (a) Let $n = q_1 = 1$ and assume in addition that F_1 has no ordinary double points. Then one has

$$\beta_1(F) \geqslant \left| 2\frac{a(d-a)}{d^2} x^2 + \sigma_{(a/d)}(K) - \operatorname{sign} M \right| - \rho_2(M).$$

This is a theorem of O. Ya. Viro [V_2]. He states the result for $d = 2$ and says there are analogous results for d a prime power.

(b) Suppose (4.1) shows that a certain configuration $\{F_i\}$ cannot arise in M with $[F_i] = x_i$. Reorient M, if necessary, so that the expression inside that absolute value sign is positive. Consider a configuration $\{F_i'\}$ identical to that above except $\{F_i'\}$ has some additional ordinary negative double points. The obstruction to realizing

x_i with $\{F_i'\}$ is identical to that for $\{F_i\}$. For example one cannot represent $3[P_1] \in H_2(P_2)$ by a smoothly immersed 2-sphere with only negative double points. This observation also follows from the arguments of §6 but perhaps would not have been noticed without the aid of (4.1).

(c) The hypothesis in (4.1) that the configuration be connected is not very restrictive. In fact any configuration may be made connected by picking paths in M between two surfaces that otherwise miss $\cup F_i$. A neighborhood of such a path is a 4-disk and $\cup F_i$ intersects the boundary of this 4-disk in a link $\mathfrak{L}$ that consists of two unknotted unlinked components. $\mu(\mathfrak{L}) = 2$ and for any well representation $\eta(\mathfrak{L}, \psi) = 1$ and $\sigma(\mathfrak{L}, \psi) = 0$.

EXAMPLES. If S^2 smoothly embeds in P_2 except for a point where it is a cone on a knot K and represents $dq[P_1] \in H_2(P_2)$ then for $0 < s < d$

$$\sigma_{(s/d)}(K) = \begin{cases} -2s(d-s)q^2 & \text{or} \\ 2 - 2s(d-s)q^2 = \sigma_{(s/d)}(K(dq, dq-1)). \end{cases}$$

Here $K(m, n)$ will denote the (m, n) torus link and to fix orientation conventions $K(2, 2)$ is the positive Hopf link. One needs to know that $\sigma_{s/d}$(knot) is even. This follows from equations (3.1), (3.6) and the definition. The identification with $\sigma_{(s/d)}(K(dq, dq-1))$ can be shown using (5.1). If $s = 0 \bmod p$ then we must use $d' = d/(s, d)$ for d in (4.2) to get the above result.

The curve $z_0 z_1^{m-1} = z_2^m$ in P_2 is an embedded 2-sphere F with a single singularity at $[1, 0, 0]$ given by the cone on $K(m, m-1)$ and $[F] = m[P_1]$. See **[H-N-K**, p. 90]. Thus letting $m = dq$ we see $K(dq, dq-1)$ can actually be realized. Which other signatures can be realized? Kervaire and Milnor **[K-M]** show how $K(2, 3)$ can appear as the singularity of a 2-sphere representing $2[P_1]$.

Now let $m = dq - 1$ instead and suppose F intersects another 2-sphere F_2 representing the generator in a single point away from $[1, 0, 0]$ and that F_2 is smooth away from its intersection with F. Let $\mathfrak{L}$ be the link that gives the intersection. It has two components with linking number m. The normal sphere bundle of one component of $\mathfrak{L}$ is a torus embedded in the complement. This embedding induces an isomorphism on $H_1(\ , Z_d)$. By an argument similar to the proof of (1.5), it follows that $\eta_{(s/d)}(\mathfrak{L}) = 0$. Then $\sigma_{(s/d)}(\mathfrak{L})$ is determined by (4.1):

$$\sigma_{(s/d)}(\mathfrak{L}) = 1 - 2sq^2(d-s) - \sigma_{(s/d)}(K(m, m-1))$$

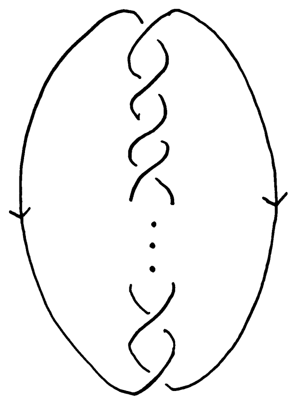
for $0 < s < d$. Here again use $d' = d/(s, d)$.

In fact F intersects the P_1 given by $z_0 = 0$ in one point $[0, 1, 0]$. The intersection is given by the cone on $\mathfrak{L}$ (see Figure 4) which is actually $K(2, 2m)$. Using (5.1) one can show

$$\sigma_{(s/d)}K(2, 2m) = 1 + 2[2s/d] - 4sq$$

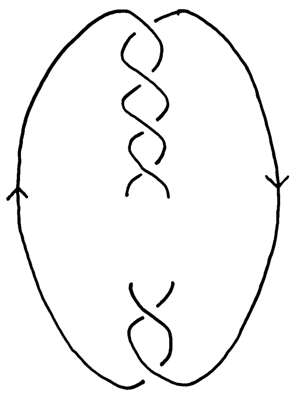
for $2s \leqslant d$. The link $\mathfrak{L}'$ (see Figure 4) has Seifert matrix $[-m]$ so $\sigma_{(s/d)}(\mathfrak{L}') = -1$. It follows that F cannot intersect another 2-sphere with intersection given by $\mathfrak{L}'$ unless $m = 1$ (in which case $\mathfrak{L} = \mathfrak{L}'$).

$\mathfrak{L} =$



m full right handed twists

$\mathfrak{L}' =$



m full left handed twists

FIGURE 4

We now derive the Tristram-Murasugi bounds for the slice genus of links. Define the genus $g(F)$ of a surface F to be the genus of the closed surface obtained by adjoining a 2-disk to each boundary component of F.

COROLLARY 4.3. *Let $\mathfrak{L}$ be a link bounding a surface F smoothly embedded in D^4 with no closed components. Then*

$$2g(F) \geqslant |\sigma_{(s/d)}(\mathfrak{L})| - \mu(\mathfrak{L}) + \beta_0(F) + |\eta_{(s/d)}(\mathfrak{L}) - \beta_0(F) + 1|.$$

PROOF. Consider the connected configuration of $\beta_0(F)$ surfaces in S^4 formed by adjoining the cone on $\mathfrak{L}$ in a second copy of D^4 and apply (4.1).

REMARKS. Murasugi **[M₁]** first proved (4.3) for $d = 2$ but he left out the expression given in the second absolute value sign. Tristram then proved (4.3) for the case $d = p$ and $s = [p/2]$ (though his proof probably works in general). The methods of both Murasugi and Tristram are very geometric and involve no mention of covers. Next Kauffman and Taylor **[K-T]** gave a proof of (4.3) in the case $d = 2$ by relating the signature and nullity to branched covers. Recently Kauffman **[Ka₂]** gave a proof of (4.3) in the case $\beta_0(F) = 1$ using the same approach as **[K-T]**.

5. Signature calculations. The main purpose of this section is to make the results of this paper (except for 2.4) independent of the G-Signature Theorem. In fact, it is an easy matter using the results of this section to prove the G-Signature Theorem for finite cyclic semifree actions on 4-manifolds with orientable fixed point set. In the spring and early summer of 1975, I found such a geometric proof, similar but not identical to that indicated below. C. McA. Gordon has found a similar proof of this case of the theorem. Also R. A. Litherland has a proof where the fixed point set is assumed to consist of orientable surfaces.

We begin by deriving Brieskorn type formulas for the signature and nullity of torus links. The formula for $\sigma_{(s/d)}$ below follows from [**Z**, Theorem 1, p. 118] and (3.4). It also appears in [**Li**].

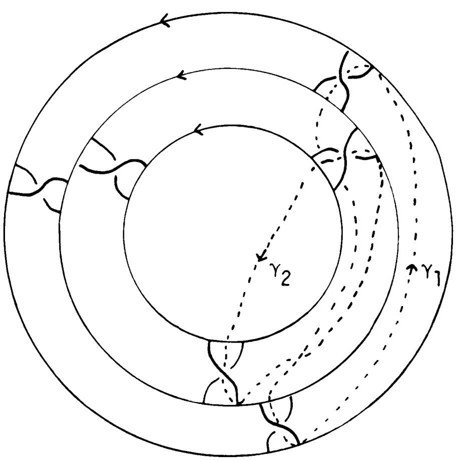


FIGURE 5

PROPOSITION 5.1.

$$\sigma_{(s/d)}(K(m,n)) = \sum_{\substack{0<i<m\\0<j<n}} \varepsilon\left(\frac{i}{m}+\frac{j}{n}+\frac{s}{d}\right),$$

$$\eta_{(s/d)}(K(m,n)) = \sum_{\substack{0<i<m\\0<j<n}} \delta\left(\frac{i}{m}+\frac{j}{n}+\frac{s}{d}\right)$$

where

$$\varepsilon(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \bmod 2 \\ 0 & \text{if } x \in \mathbf{Z} \\ -1 & \text{if } 1 < x < 2 \bmod 2 \end{cases} \quad \text{and} \quad \delta(x) = \begin{cases} 1, & x \in \mathbf{Z}, \\ 0, & x \notin \mathbf{Z}. \end{cases}$$

PROOF. We will use the Seifert surface F pictured in Figure 5. There are m concentric disks lying over one another and $(m-1) \times n$ connecting bands. Let T be the symmetry of F given by rotating F through an angle $2\pi/n$. Let $\gamma_1, \ldots, \gamma_{m-1}$ be the curves indicated by the dotted lines in Figure 5. $\{T^j\gamma_i\}$ for $0 < i < m$ and $0 < j < n$ with the lexigraphical ordering (beginning $T^1\gamma_1, T^2\gamma_1, \ldots$) forms a basis $\mathfrak{B}$ for $H_1(F)$. With respect to $\mathfrak{B}$, the Seifert form is given by $-\Lambda_n \otimes \Lambda_m$, and the map T is given by $T_n \otimes I_{m-1}$ where

$$\Lambda_n = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \bigcirc \\ & & 1 & -1 & \\ & \bigcirc & & & \ddots \end{bmatrix}_{n-1 \times n-1},$$

$$T_n = \begin{bmatrix} 0 & & & & -1 \\ & & \bigcirc & & \\ 1 & 0 & & & -1 \\ & & & & \vdots \\ & 1 & 0 . & & \\ & & & \cdot 0 & -1 \\ & \bigcirc & & & \\ & & & 1 & -1 \end{bmatrix}_{n-1 \times n-1}.$$

This suggests the observation that T_n preserves the form given by Λ_n. To see this visually let $m = 2$. T_n has eigenvalues $\{\lambda^1, \ldots, \lambda^{n-1}\}$ where $\lambda = e^{(2\pi i/n)}$. So the eigenvectors of v_i (determined up to a scalar factor) diagonalize Λ_n as a sesquilinear form. In fact there is a basis of eigenvectors with respect to which the form is given by

$$\Delta_n = \begin{bmatrix} 1-\lambda & & & \\ & 1-\lambda^2 & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & 1-\lambda^{n-1} \end{bmatrix}.$$

See **[D-K]** or **[Ka$_2$]**.

So $-\Delta_n \otimes \Delta_m$ is the matrix for the Seifert form (over $\mathbf{C}$). Then $(-\Delta_n \otimes \Delta_m)_{(s/d)}$ is also diagonal with entries on the diagonal $\{d_{ij}\}$ for $0 < i < m$ and $0 < j < n$ where

$$d_{ij} = -2 \text{ Real part}(1-\lambda^i)(1-\beta^j)(1-\omega^s) \quad \text{and} \quad \beta = e^{2\pi i/m}.$$

It is easy to see that d_{ij} is positive, negative or zero accordingly as

$$\varepsilon(i/m + j/n + s/d)$$

is $+1$, -1, or zero. This completes the proof.

We now explain the connection between $\sigma(L, \psi)$ and the Atiyah-Singer α-invariant. Let T act with finite order on a $2k$-manifold M. Atiyah and Singer **[A-S**, p. 578] define a complex number sign(T, M) (the T-signature). Specify that T acts on the homology preserving the intersection pairing (as opposed to cohomology) to get correct signs. Now suppose T acts freely with order d on a $2k-1$ manifold N. Then by arguments in §1 $r(T, N)$ bounds a free action on some $2k$-manifold (T, M). Then define

$$\alpha(T, N) = (1/r)\text{sign}(T, M).$$

One can show

$$\operatorname{sign}(T, M) = \sum_{j=0}^{d-1} \omega^j \sigma_j(M).$$

It follows that (see **[C-G₁**, p. 6])

$$\alpha(T, L_\psi) = \sum_{s=1}^{d-1} \omega^s \sigma(L, s\psi)$$

and

$$\sigma(L, \psi) = \frac{1}{d} \sum_{s=1}^{d-1} (\omega^{-s} - 1)\alpha(T^s, L_\psi).$$

Let B be the d-fold branched cover of S^2 along d points given by ψ: ($S^2 - d$ points) $\to \mathbf{Z}_d$ where ψ maps the "meridian" of each point to one. The following proposition was probably first proved directly by Erich Ossa **[O]** in his proof of the G-Signature Theorem for finite groups. It is given as an unproved axiom in **[G]**.

PROPOSITION 5.2. *$\sigma_j(B) = 2j - d$ and $\operatorname{sign}(T^j, B) = d(\omega^j + 1)/(\omega^j - 1)$ for $0 < j < d$.*

PROOF. Consider the surface F used in the proof of (5.2) and let $m = n = d$. T acts on F with d fixed points. A fundamental domain for this action is cut out by a pie-shaped region of space with angle $2\pi/d$. It is easy to see that F is the d-fold cover of a 2-disk along d points. By Novikov additivity we only need to calculate $\sigma_j(F)$.

The intersection pairing on F is given by $(-\Lambda_d \otimes \Lambda_d) - (-\Lambda_d \otimes \Lambda_d)^T$ with respect to $\mathfrak{B}$. Here we are using a well-known relation between the intersection pairing and the Seifert pairing. This form restricted to the ω^j eigenspace is given by the matrix $-(1 - \omega^j)\Delta_d + (1 - \bar{\omega}^j)\Delta_d^*$. The associated hermitian pairing is given by the matrix $-i(1 - \omega^j)\Delta_d + i(1 - \bar{\omega}^j)\Delta_d^*$. This has entries down the diagonal $\{-2 \operatorname{Real} i(1 - \omega^j)(1 - \omega^k)\}$ for $0 < k < d$. Thus

$$\sigma_j(F) = - \sum_{0<k<d} \varepsilon((j + k)/d) = 2j - d.$$

The second formula follows easily.

The derivation in **[C-G₁]** of the formula for $\sigma(L(m, n), \chi)$ given in Example 3.9 and used in the proof of (3.8) requires the following formula for the α-invariant of free orthogonal actions on S^3.

PROPOSITION 5.3. *Let T act on $\mathbf{C}^n$ by $T(z_1, \ldots, z_n) = (\omega^{j_1} z_1, \ldots, \omega^{j_n} z_n)$ where $(j_k, d) = 1$. Let S^{2n-1} be the unit sphere in $\mathbf{C}^n$ oriented as the boundary of the unit disk*

$$\alpha(T, S^{2n-1}) = - \prod_{k=1}^{n} \frac{\omega^{j_k} + 1}{\omega^{j_k} - 1}.$$

PROOF. Consider $\prod_k (T^{j_k}, B)$. This is a closed $\mathbf{Z}_d$ manifold with d^n isolated fixed points. The action at each of these points is that given above. The multiplicative property of the G-signature gives the result.

PROPOSITION 5.4. *Let N be a closed oriented 2-disk bundle over a closed connected surface F with self-intersection dq. Let $L = \partial N$ and $\psi \in H^1(L, \mathbf{Z}_d)$ with $\delta\psi$ Lefschetz dual to $a[F]$ with $0 < a < d$ and $(a, d) = 1$. Then*

$$\sigma(L, \psi) = \frac{2(d-a)}{d} aq - \frac{q}{|q|}$$

(*interpret* $0/|0| = 0$).

PROOF. We first remark that given F, dq, and a, any such cover L is equivalent to the action on the S^1 bundle over F with Euler class q given multiplication by ω^a.

We only need to consider $q \geqslant 0$. (If $q \leqslant 0$, reverse the orientation of the bundle.) Suppose first $F = S^2$ and $q = 1$, so

$$\sigma(L, \psi) = \sigma(-L(d, 1), \chi^a) = 4(\text{Int}\,\Delta - \text{Area}\,\Delta) = 2(d-a)a/d - 1.$$

Here Δ has vertices $(0, 0)$, $(a, 0)$, $(a, a/d)$. We will refer to this cover as L'_ψ. The above calculation can also be done directly using (5.3).

Let W be a bordism between F and q copies of S^2 and $x \in H^2(W)$ be Lefschetz dual to a collection of q paths in W each joining F to a different S^2. Let Q and R be the S^1 bundles over W with Euler classes x and dx. Q is a d-fold cover of R. If T: $Q \to Q$ is given by multiplication by ω^a, then $\partial(Q, T) = L_\psi - qL'_\psi$. Since multiplication by ω^a is homotopic to the identity, $\sigma_1(Q) = 0$. Since the signature of the boundary of the disk bundle of R is zero, $\text{sign}(R) = (q-1)q/|q|$. The result now follows easily.

REMARK. We can now outline a very geometric proof that $\Omega_3(B\mathbf{Z}_d)$ is torsion. In the proof of (3.6), an explicit branched cover of a 4-manifold is given extending a given free $\mathbf{Z}_d$ action on a 3-manifold. This provides a bordism of free $\mathbf{Z}_d$ actions to one of the type considered in (5.4). The proof of (5.4) gives a rational bordism to actions on 3-spheres. The proof of (5.3) gives a rational null bordism for these actions.

6. Ad hoc geometric arguments and Rokhlin's Theorem. We consider the situation of Theorem 0.1. Namely we have a collection of n smoothly embedded surfaces F_i in general position in a closed 4-manifold M. Moreover we have a relation $\sum a_i x_i = dy$ where $x_i = [F_i] \in H_2(M, \mathbf{Z})$. We would like to say something about $\beta_1(F_i)$ and α_{ij}, the geometric number of intersections between F_i and F_j, using only Rokhlin's Theorem (namely Theorem 0.1 for $n = 1$).

The basic idea is to use the surfaces F_i to construct a single connected F representing dy in M and express $\beta_1(F)$ in terms of $\beta_1(F_i)$ and α_{ij}. One can then apply Rokhlin's Theorem and get a lower bound on $\beta_1(F)$. I first became aware of this possibility in spring 1974 and gave a seminar talk on what this said about the homology class $(2, 3)$ in $S^2 \times S^2$. Just recently S. Weintraub has informed me that he has made a similar observation **[We]**.

The construction of F goes as follows. Begin by pushing off a_i copies of each F_i, creating a configuration of $\sum a_i$ surfaces in M with only Hopf link singularities. One then may resolve such singularities by removing two 2-disks, one from each sheet of an intersection, and replacing these disks with a single cylinder. One can do this since both Hopf links bound cylinders in D^4 (actually in S^3). If the two sheets that intersect belong to the same surface, this process raises $\beta_1(F)$ by two. Otherwise the two surfaces are joined to form a single surface with Betti number the sum of the original Betti numbers.

One could proceed in this manner and eventually end up with a connected surface F representing $\Sigma\, a_i x_i$. However if at any point in this process a connected surface G possesses both a positive and negative intersection point, there is a way to resolve this in a way that is less "expensive" in terms of the final $\beta_1(F)$. Pick a path γ on G running between these two intersection points and missing all other intersection points and look at the normal S^1 bundle of G in M restricted to γ. This is a cylinder which may be used to resolve two intersection points at once.

Given a relation $\Sigma\, a_i x_i = 0 \bmod d$ with $0 < a_i < d$, there are many related relations obtained by changing each a_i by the same scalar factor. One may also reverse the orientation F_i and change a_i for $(d - a_i)$. To get the best information, one should choose $[F]$ wisely. For example, if all the a_i are equal, one should construct F to represent $\Sigma\, x_i$.

We illustrate this procedure with an example discussed in the introduction where F_1 and F_2 are 2-spheres representing $x_1 = (0, 1, 0, 0)$ and $x_2 = (2m + 1, 3m + 2, 0, 0)$ in $H_2(S^2 \times S^2 \# S^2 \times S^2)$ and $2m + 1$ is a prime.

Suppose F_1 intersects F_2 at a positive and b negative points. So $a + b = \#$ and $a - b = 2m + 1$. Construct F representing $2m + 1(1, 2, 0, 0)$ by pushing off m copies of F_1 and tubing to F_2. First tube each copy of F_1 to F_2 at a positive intersection point. One then has an immersed 2-sphere with $(a - 1)m$ positive double points and bm negative double points. One then removes cancelling pairs of double points until one has an immersed surface with $\beta_1 = 2bm$ with $(a - b - 1)m$ positive double points. Finally resolve these to get F with $\beta_1(F) = 2bm + 2(a - b - 1)m = m(\# + 2m - 1)$. By Rokhlin's Theorem (choosing $a_1 = m$), $\beta_1(F) \geqslant 8m^2 + 8m - 4$. So $\# \geqslant 6m + 9 - (4/m)$.

If not all the a_i are equal, then one will never get an expression for $\beta_1(F)$ involving $\Sigma\, \beta_1(F_i)$. Thus one cannot hope to derive Theorem 0.1 in this way. Suppose now all the $a_i = a$ and $(2(d - a)a/d^2)(\Sigma\, x_i)^2 \geqslant \Sigma_{i<j}\, x_i x_j + \text{sign}\, M$. Assume moreover that there is a sequence of $(n - 1)$ positive intersection points between the n surfaces F_i, such that, after resolving each of these surfaces to form an immersed surface G, G is connected. This last condition may be guaranteed by various conditions involving $x_i x_j$ including the one given in the introduction. A surface G so constructed will have $\beta_1(G) = \Sigma\, \beta_1(F_i)$ and k positive double points and l negative double points where $k + l = \# - (n - 1)$ and $k - l = \Sigma\, x_i x_j - (n - 1)$. If $k \geqslant l$, then we can derive the conclusion of Theorem 0.1 by the above "tubing" procedure.

To get this same conclusion in general, we need to introduce another technique for resolving double points. We can "blow up" a positive (respectively negative) double point by replacing a 4-ball neighborhood of this point with a punctured P_2 (respectively $\overline{P}_2$). The original surface may then be extended to an embedding in the blown-up part of the manifold. One sheet will cross $P_1 \subset P_2$ positively and one negatively. If we blow up all the double points of G in the above example, we will get a surface F with $\beta_1 F = \sum \beta_1(F_i)$ embedded in $M' = M \#^k P_2 \#^l \bar{P}_2$ representing $(\sum x_i, 0, \ldots, 0) \in H_2(M')$. We have sign $M' =$ sign $M + \sum_{i<j} x_i x_j - (n-1)$ and $\rho_2(M') = \rho_2(M) + \# - (n-1)$. Applying Rokhlin's Theorem now yields

$$\beta_1(F) \geqslant \left|(2(d-a)a/d^2)\left(\sum x_i\right)^2 - \text{sign } M'\right| - \rho_2(M').$$

This is also the conclusion of Theorem 0.1. The blowing-up construction also gives a geometric explanation of the phenomena discussed in Remark (b) following (4.2).

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